The thorianite used in this experiment contained initially 9 c.c. of helium per gramme. The annual rate of production per gramme is, from the experiment, certainly less than $3{\cdot}7 \times 10^{-8}$ c.c. It follows that the accumulation cannot have taken less than 240 million years. It may have taken considerably more.

In this account, which is of an interim character, I have purposely avoided detailed description of experimental methods or discussion of the results in their geological bearing. These matters are deferred.

The Aberrations of a Symmetrical Optical Instrument.

By H. C. Pocklington, M.A., D.Sc., F.R.S.

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1. Attention has been called by Rayleigh in a paper cited below to the ease with which the general nature of the aberrations of optical instruments can be discussed by means of a modified characteristic function. The object of the present paper is to work this method in more detail in order to get a complete account of the aberrations and to connect it with the known method of calculating the aberrations of a particular system.

We write down the doubly modified function in § 2, and in § 3 deduce from it the singly modified function used in the rest of the paper. The meanings of the coefficients are given in § 4. Attention having been called in § 5 to some features that limit the usefulness of the method, a new expansion of more practical value is found in § 6. The meanings of the coefficients of this expression are given in detail in § 7, and some relations between them (*i.e.* between the spherical aberration, coma, etc.) in § 8. The numerical calculation is discussed in § 9. In § 10 another method of calculating the characteristic function is sketched out.

2. The characteristic function, strictly so called, is the integral of μ taken along the path of a ray of light from a point (say x, y, z) in the initial medium to another (say x', y', z') in the final medium, the integral being expressed in terms of these six co-ordinates. Supposing the initial and final media to have refractive index unity, we have

$$dV/dx = l, \text{etc.}, \ dV/dx' = l', \text{etc.},$$

where l, m, n, l', m', n' are the direction cosines of the ray in the two media, and V satisfies the two partial differential equations

$$(dV/dx)^2+(dV/dy)^2+(dV/dz)^2 = 1, \text{ and}$$

$(dV/dx')^2+(dV/dy')^2+(dV/dz')^2=1$. If we write $U \equiv V - lx - my$, the variation of U is $dU = l'dx'+m'dy'+n'dz'-xdl-ydm+ndz$. Hence if U is expressed as a function of l, m, z, x', y', z', its partial differential coefficients with respect to the variables are $-x$, $-y$, n, l', m', n', so that U satisfies the same partial differential equation with respect to x', y', z' that V does, and also satisfies $(dU/dz)^2+l^2+m^2=1$.

If we write $T \equiv V-lx-my-l'x'-m'y'$ and express it as a function of l, m, z, l', m', z', its partial differential coefficients with respect to these variables are $-x$, $-y$, n, $-x'$, $-y'$, n' and it satisfies the equations

$$(dT/dz)^2+l^2+m^2 = 1 \quad \text{and} \quad (dT/dz')^2+l'^2+m'^2 = 1.$$

Hence we have

$$T = z\sqrt{(1-l^2-m^2)}+z'\sqrt{(1-l'^2-m'^2)}+\phi\,(l, m, l', m'),$$

where z and z' are measured in opposite directions. Expanding the arbitrary function as far as terms of the fourth order, and remembering that in a symmetrical instrument ϕ can only be a function of l^2+m^2, $ll'+mm'$, $l'^2+m'^2$, we have

$$\begin{aligned} T = {} & z+z'+\tfrac{1}{2}\{(a-z)(l^2+m^2)+zf(ll'+mm')+(b-z')(l'^2+m'^2)\} \\ & -\tfrac{1}{4}\{(\alpha+\tfrac{1}{2}z-\tfrac{1}{2}a)(l^2+m^2)^2+\gamma(l^2+m^2)(ll'+mm')+(\epsilon-\eta)(ll'+mm')^2 \\ & +\eta(l^2+m^2)(l'^2+m'^2)+\delta(ll'+mm')(l'^2+m'^2) \\ & +(\beta+\tfrac{1}{2}z'-\tfrac{1}{2}b)(l'^2+m'^2)^2\}. \end{aligned}$$

The quantities α, β, γ, δ, ϵ, η determine, as we shall see, the aberrations of the image for all positions of the object, and hence may be called the coefficients of aberration of the system.

3. This function is not convenient for general use and accordingly we deduce the value of U, which is $T+l'x'+m'y'$, expressed in terms of x, y, z, l', m', z'. Put $z-a=\zeta$, $z'-b=\zeta'$ and let the quartic terms in T be $-W$. Then

$$x' = -dT/dl' = \zeta'l'-fl+dW/dl',$$
$$y' = -dT/dm' = \zeta'm'-fm+dW/dm',$$

and

$$\begin{aligned} U & = T-l'dT/dl'-m'dT/dm' \\ & = z+z'+\tfrac{1}{2}\zeta'(l'^2+m'^2)-\tfrac{1}{2}\zeta(l^2+m^2)-W+l'dW/dl'+m'dW/dm' \\ & = z+z'+(\zeta'l'+dW/dl')^2/2\zeta'+(\zeta'm'+dW/dm')^2/2\zeta'-\tfrac{1}{2}\zeta(l^2+m^2)-W, \end{aligned}$$

neglecting terms of the sixth order of small quantities. We now substitute the accurate values of l' and m' in the second order terms, and the approximate values $(x'+fl)/\zeta'$, $(y'+fm)/\zeta'$ in W, and expand it in powers and products of $x'^2+y'^2$, $lx'+my'$, and l^2+m^2, obtaining

$$\begin{aligned}
\mathrm{U} = {} & z+z'+(x'+fl)^2/2\zeta'+(y'+fm)^2/2\zeta'-\tfrac{1}{2}\zeta(l^2+m^2) \\
& -\tfrac{1}{4}[(x'^2+y'^2)^2(\beta+\tfrac{1}{2}\zeta')/\zeta'^4 \\
& +(x'^2+y'^2)(lx'+my')\{4(\beta+\tfrac{1}{2}\zeta')f/\zeta'^4+\delta/\zeta'^3\} \\
& +(lx'+my')^2\{6(\beta+\tfrac{1}{2}\zeta')f^2/\zeta'^4+3\,\delta f/\zeta'^3+\epsilon/\zeta'^2\} \\
& +(mx'-ly')^2\{2(\beta+\tfrac{1}{2}\zeta')f^2/\zeta'^4+\delta f/\zeta'^3+\eta/\zeta'^2\} \\
& +(lx'+my')(l^2+m^2)\{4(\beta+\tfrac{1}{2}\zeta')f^3/\zeta'^4+3\,\delta f^2/\zeta'^3+2\epsilon f/\zeta'^2+\gamma/\zeta'\} \\
& +(l^2+m^2)^2\{(\beta+\tfrac{1}{2}\zeta')f^4/\zeta'^4+f^3/\zeta'^3+\epsilon f^2/\zeta'^2+\gamma f/\zeta+(\alpha+\tfrac{1}{2}\zeta)\}].
\end{aligned}$$

We can also obtain U directly from its differential equation, but the process is more troublesome than that given here.

4. Now $x = -d\mathrm{U}/dl = (\zeta\zeta'-f^2)\,l/\zeta-x'f/\zeta'$, neglecting terms of the third order. Hence f is the focal length (considered to be positive in the case of a converging system such as a convex lens) and $z = a$, $z' = b$ give the foci.

The remaining terms determine the aberration, and comparing with Rayleigh's form* we see that the coefficient of $(l^2+m^2)^2$ in U gives the spherical aberration, the preceding coefficient the coma, etc. The conditions for a perfect image are

$$\alpha+\tfrac{1}{2}\zeta = -\gamma\zeta/4f = \epsilon\zeta^2/6f^2 = -\delta\zeta^3/4f^3 = (\beta+\tfrac{1}{2}\zeta')\,\zeta^4/f^4 = \eta\zeta^2/2f^2.$$

Hence a symmetrical system cannot be free from defect unless the object and image are equidistant from their foci. These equations do not give the values of α, β, γ, δ, ϵ, η, hence the aberrations for other positions of the object are not determinate. The spherical aberration involves all the constants except η, and hence if it is known at five points on the axis we can find these five constants and completely determine the aberrations in the principal plane. The value of η is not determinable thus, but we shall see that it can be very easily found by another method. The method here suggested for calculating the aberrational coefficients of the system is, however, in practice, lengthy.

5. There is some artificiality about the aberrations (except the spherical aberration) as defined above, for if we reverse the direction of the light, thus interchanging the object and image, the new aberrations not only differ from the old, but (except the spherical aberration) are not simply related to them. This is specially striking in the case of the distortion, which in one case

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* Lord Rayleigh, "Hamilton's Principle and the Five Aberrations of von Seidel," 'Phil. Mag.,' Ser. 6, vol. 15 (1908), p. 682.

gives rise, when $y = 0$, to a term $(\beta+\frac{1}{2}\zeta')f/\zeta'^4+\delta/4\zeta'^3$ in x, and in the other case to a term $(\alpha+\frac{1}{2}\zeta)f/\zeta^4+\gamma/4\zeta^3 = (\alpha+f^2/2\zeta')\zeta'^4/f^7+\gamma\zeta'^3/4f^6$.

It is clear that one may be zero and the other not, yet the actual distortions must be in a ratio equal to the magnification. The reason for this discrepancy is that the position of the image is considered to be the point where a ray of light coming from the object meets the image-plane at right angles, so that the rays are not the same in the two cases. Moreover, it will often happen that towards the margin of the field there is no such ray, so that the point that serves to define the distortion has no physical existence.

6. The rays that pass through the instrument are limited by some diaphragm (which may be the mount of some lens of the system). The image of this formed by the refracting surfaces that follow it is called the exit pupil, and that formed by the preceding surfaces is called the entrance pupil (so that the two pupils are images of each other with respect to the whole system). Let the exit pupil be at $\zeta = \xi$. Put $x+\delta x$ for x, $y+\delta y$ for y, where x, y is the point where the image would be if there were no aberration, so that $x = x'\zeta/f$, $y = y'\zeta/f$. The ray through the centre of the exit pupil is given by $l = x/(\zeta-\xi)$, $m = y/(\zeta-\xi)$. Therefore put $l-x/(\zeta-\xi) = \lambda$, $m-y/(\zeta-\xi) = \mu$. Substituting in U, expanding, and putting

$$x' = xf/\zeta,\ y' = yf/\zeta,\ \zeta' = f^2/\zeta,$$

we find the quartic terms to be

$$\begin{aligned}
&-\tfrac{1}{4}[(\lambda^2+\mu^2)\{(\alpha+\tfrac{1}{2}\zeta)+\gamma\zeta/f+\epsilon\zeta^2/f^2+\delta\zeta^3/f^3+(\beta+\tfrac{1}{2}\zeta')\zeta^4/f^4\}\\
&+(\lambda^2+\mu^2)(\lambda x+\mu y)\{4(\alpha+\tfrac{1}{2}\zeta)+\gamma(3\zeta+\xi)/f+2\epsilon\zeta(\zeta+\xi)/f^2+\delta\zeta^2(\zeta+3\xi)/f^3\\
&\qquad\qquad -4(\beta+\tfrac{1}{2}\zeta')\zeta^3\xi/f^4\}/(\zeta-\xi)\\
&+(\lambda x+\mu y)^2\{6(\alpha+\tfrac{1}{2}\zeta)+3\gamma(\zeta+\xi)/f+\epsilon(\zeta^2+4\zeta\xi+\xi^2)/f^2+3\delta\zeta\xi(\zeta+\xi)/f^3\\
&\qquad\qquad +6(\beta+\tfrac{1}{2}\zeta')\zeta^2\xi^2/f^4\}/(\zeta-\xi)^2\\
&+(\lambda y-\mu x)^2\{2(\alpha+\tfrac{1}{2}\zeta)+\gamma(\zeta+\xi)/f+2\epsilon\zeta\xi/f^2+\delta\zeta\xi(\zeta+\xi)/f^3\\
&\qquad\qquad +2(\beta+\tfrac{1}{2}\zeta')\zeta^2\xi^2/f^4+\eta(\zeta-\xi)^2/f^2\}/(\zeta-\xi)^2\\
&+(\lambda x+\mu y)(x^2+y^2)\{4(\alpha+\tfrac{1}{2}\zeta)+\gamma(\zeta+3\xi)/f+2\epsilon\xi(\zeta+\xi)/f^2+\delta\xi^2(3\zeta+\xi)/f^3\\
&\qquad\qquad +4(\beta+\tfrac{1}{2}\zeta')\zeta\xi^3/f^4\}/(\zeta-\xi)^3\\
&+(x^2+y^2)^2\{(\alpha+\tfrac{1}{2}\zeta)+\gamma\xi/f+\epsilon\xi^2/f^2+\delta\xi^3/f^3+(\beta+\tfrac{1}{2}\zeta')\xi^4/f^4\}/(\zeta-\xi)^4].
\end{aligned}$$

7. Let the coefficients of $(\lambda^2+\mu^2)^2$, etc., inside the [ ] be called A, B, C_1, C_2, D, E, respectively. Then putting $y' = 0$ for simplicity (from the symmetry of the system there is no loss of generality), and remembering that $d/dl = d/d\lambda$, $d/dm = d/d\mu$, we find

$$\delta x = A\lambda(\lambda^2+\mu^2)+B(3\lambda^2+\mu^2)x/4+C_1\lambda x^2/2+Dx^3/4,$$

$$\delta y = A\mu(\lambda^2+\mu^2)+B\lambda\mu x/2+C_2\mu x^2/2.$$

Hence A is the coefficient of spherical aberration. The lateral aberration of the extreme rays (those passing by the rim of the exit pupil) is $A\rho^3$, where ρ is the angular radius of the exit pupil as seen from the centre of the image. The radius of the circle of least confusion is $A\rho^3/4$. The aberration is considered to be positive when the caustic points in the direction of the light. The coefficient has the dimensions of length.

The quantity B is the coefficient of coma. If the other aberrations are absent, the length of the balloon-shaped image of a point is $3B\rho^2x/4$, the radius of the circle made by the extreme rays is $B\rho^2x/4$, and the distance of its centre from the image formed by the central ray is $B\rho^2x/2$. The coma is positive if the faint body of the balloon is farther from the axis than the bright point that suggests the car. This coefficient has the dimensions of a number.

The quantities C_1, C_2 are the coefficients of primary and secondary curvature respectively. If the pupil is so narrow that coma and spherical aberration can be neglected (otherwise there is nothing to call an image), the curvature of the primary image is C_1 and that of the secondary image C_2. The curvatures are considered positive if concave to the incident light. The difference C_1-C_2 may be called the coefficient of astigmatism. All these coefficients have the dimensions of 1/length.

The quantity D is the coefficient of distortion. The displacement of the image formed by the central ray from its correct position is $Dx^3/4$. The distortion is considered to be positive if the displacement is outwards (pincushion distortion). The coefficient has the dimensions of $1/(\text{length})^2$.

The quantity $C_3 = (3\,C_2-C_1)/2$, which is a weighted mean of the primary and secondary curvatures, the weights being -1 and 3, is equal to $(3\eta-\epsilon)/2f^2$, and hence is independent of the position of the image and pupil. If the final medium has a refractive index μ, we have only to put μx, μy, $\mu\zeta$ for x, y, ζ in the expression for U. Hence the same statement still holds. We may call C_3 the Coddington-Petzval curvature. Now let the object be at the first refracting surface and the pupil at the vertex of this surface, then for the first image $C_3/\mu_1 = (1-1/\mu_1)\,\rho_1$, where ρ_1 is the curvature of the surface and $1, \mu$ the refractive indices of the first and second media. Now shift the object to such a place that the image falls on the second refracting surface, and shift the pupil to the vertex of this surface. Then the first image still has $C_3/\mu_1 = (1-1/\mu_1)\,\rho_1$, and the second image has

$$C_3/\mu_2 = (1-1/\mu_1)\,\rho_1 + (1/\mu_1-1/\mu_2)\,\rho_2,$$

and so on. Hence the Coddington-Petzval curvature of the final image is $C_3 = \Sigma\,(1/\mu_{n-1}-1/\mu_n)\,\rho_n$, which, if the system consists of thin lenses, can be written $C_3 = \Sigma 1/\mu_n f_n$. We note that no assumption as to the shape of the refracting surfaces has been made except that they are surfaces of revolution. The importance of this result is very great, as it reduces the problem of determining the aberrations of a system to that of determining the aberrations of rays lying in a principal plane.

The quantity E does not appear in the expressions for the aberrations, but is connected with the excess of the time taken by the light in passing from a point in the object to its image over that taken when the object is on the axis.

These quantities may be called the coefficients of aberration of the image. If we put the exit pupil at infinity, we get the coefficients of § 2.

8. These coefficients satisfy the equations

$$(\zeta-\xi)^2 d\mathrm{E}/d\xi = \mathrm{D}, \quad (\zeta-\xi)^2 d\mathrm{D}/d\xi = 2\,\mathrm{C}_1, \quad (\zeta-\xi)\, d\mathrm{C}_1/d\xi = 3\,\mathrm{B},$$
$$(\zeta-\xi)^2\, d\mathrm{B}/d\xi = 4\,\mathrm{A},$$

so that if the distortion vanishes for four (and therefore for all) positions of the exit pupil, the other aberrations in the principal plane also vanish. If it vanishes for three consecutive positions of the pupil, the only defect of the primary image is spherical aberration, which, if its coefficient is not too great, is of comparatively small importance in a photographic lens (because ρ is smaller than l), so that if the secondary curvature also vanishes the lens will be a good one. We also see that a similar statement holds with respect to the astigmatism. If the astigmatism vanishes for two consecutive positions of the pupil the coma also vanishes, so that we may either make the pupil rather large or disregard moderate inaccuracies in its position.

We notice another close relationship between B and D. If we denote by B′ the value that B assumes when the image and pupil are interchanged, we have

$$\begin{aligned}\mathrm{D} &= -\mathrm{B}'/(\zeta-\xi)^2 + 2\,(f^2-\xi^2)/f^2\,(\zeta-\xi)^2\\ &= -\mathrm{B}'/(\zeta-\xi)^2 + 2\,(\xi'-\xi)/\xi'\,(\zeta-\xi)^2.\end{aligned}$$

We easily see that a relationship of this kind should hold, for the distortion depends on the tangent of the angle that the ray on emergence from the exit pupil makes with the axis, and the coma, when the image is made to occupy this place, depends on the sine of the same angle.

9. To calculate the values of these coefficients for a particular system, we may follow the method given by Whittaker.* On going through his working and replacing the omitted factors, we get the $\delta x/x$ of our notation, or $\delta l/l$ of his, and dividing by $l^2/4$, we have

$$\mathrm{D} = 2\,(h_n/v_n)^2 \sum_1^n \frac{\mathrm{Z}_p}{h_p{}^2\mathrm{W}_p\,(\mathrm{Z}_p-\mathrm{W}_p)} \left\{\frac{\mathrm{Z}_p{}^2}{(\mathrm{Z}_p-\mathrm{W}_p)^2}\left(\frac{1}{\mu_{p-1}u_p}-\frac{1}{\mu_p v_p}\right)+\frac{1}{r_p}\left(\frac{1}{\mu_p}-\frac{1}{\mu_{p-1}}\right)\right\}.$$

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* E. T. Whittaker, 'The Theory of Optical Instruments,' p. 41.

In this expression r_p is the radius of the p'th refracting surface, considered to be positive if the surface is convex to the incident light, and μ_{p-1}, μ_p are the refractive indices of the media on either side of it. The distances of the intermediate object and image relative to the surface from the surface are u_p and v_p, considered to be positive when measured from the surface in the direction of the light, so that $\mu_{p-1}(1/r_p - 1/u_p) = \mu_p(1/r_p - 1/v_p)$. Each of these equal quantities is called W_p. The quantities x_p, y_p, and z_p are similarly related to the images of the pupil. Also h_p is the height at which a ray from the point of the object on the axis meets the p'th surface, the ray being supposed to pass so near to the axis that cubes of small quantities can be neglected.

Now*
$$\frac{1}{h_p^2(\mathrm{Z}_p - \mathrm{W}_p)} = \frac{1}{h_n^2(\mathrm{Z}_n - \mathrm{W}_n)} - \sum_p^{n-1} \frac{c_q}{\mu_q h_q h_{q+1}},$$
where the c's are the distances between the surfaces,
$$= -\frac{v_n(v_n + \xi - \zeta)}{h_n^2(\zeta - \xi)} - \Sigma,$$
since $v_n - y_n = \zeta - \xi$. Hence
$$(\zeta - \xi)^2 \frac{d}{d\xi} \frac{\mathrm{Z}_p}{\mathrm{Z}_p - \mathrm{W}_p} = -\frac{v_n^2 h_p^2}{h_n^2} \mathrm{W}_p,$$
since the terms under Σ are independent of ξ.

Hence we find immediately
$$\mathrm{C}_1 = -\sum_1^n \left\{ 3\left(\frac{\mathrm{Z}_p}{\mathrm{Z}_p - \mathrm{W}_p}\right)^2 \left(\frac{1}{\mu_{p-1}u_p} - \frac{1}{\mu_p v_p}\right) + \frac{1}{r_p}\left(\frac{1}{\mu_p} - \frac{1}{\mu_{p-1}}\right) \right\},$$
$$\mathrm{B} = 2\frac{v_n^2}{h_n^2} \sum_1^n h_p^2 \frac{\mathrm{Z}_p \mathrm{W}_p}{\mathrm{Z}_p - \mathrm{W}_p} \left(\frac{1}{\mu_{p-1}u_p} - \frac{1}{\mu_p v_p}\right),$$
$$\mathrm{A} = -\tfrac{1}{2} \frac{v_n^4}{h_n^4} \sum_1^n h_p^4 \mathrm{W}_p^2 \left(\frac{1}{\mu_{p-1}u_p} - \frac{1}{\mu_p v_p}\right).$$

Using the result at the end of § 8, we have the alternative form of D,
$$\mathrm{D} = \frac{2}{(v_n - y_n)^2} \frac{y_n^2}{k_n^2} \Sigma k_p^2 \frac{\mathrm{Z}_p \mathrm{W}_p}{\mathrm{Z}_p - \mathrm{W}_p} \left(\frac{1}{\mu_{p-1}x_p} - \frac{1}{\mu_p y_p}\right) + 2(f^2 - \xi^2)/f^2(v_n - y_n)^2,$$
where the k's are the heights at which a paraxial ray passing through the pupil and its image meets the surfaces.

10. There is another method for calculating the coefficients of aberration of a particular system. The characteristic function is the integral of μ taken along the actual ray from a point of the object to its image. Now this ray is difficult to find, but if we calculate the path of the ray, assuming that there is no aberration at any of the surfaces (which is easily done), we have a ray

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* Whittaker, *loc. cit.*, p. 27.

that is distant from the actual ray by distances of the third order. If we calculate the integral for this ray, the error vanishes to the third order (from the minimum property of V) and the next order that enters is the sixth (provided that the inclination of the ray is correct to the first order, a point that requires attention in dealing with thin lenses), so that the value of V thus calculated is correct to the fourth order. To find α, β, γ, δ, ϵ we most conveniently use a ray in the principal plane and express it in terms of the initial and final inclinations. On calculating the terms of the fourth order in V and equating coefficients, we have α, β, γ, δ, ϵ. This method appears to have no advantages except, perhaps, in the case of the mirror or simple lens.

The Boiling Point of Sulphur corrected by Reference to New Observations on the Absolute Expansion of Mercury.

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The very accurate observations* of Eumorfopoulos on the boiling point of sulphur with an air thermometer of the type described by Callendar† gave a final value 443°·58 C., and appeared to cast some doubt on the value previously assumed, namely 444°·53 C. on the scale of the constant-pressure air thermometer. It was pointed out, however, by Eumorfopoulos, and also in a note appended by Callendar,‡ that the result depended on the assumption of Regnault's results as recalculated by Broch for the absolute expansion of mercury, and that the final value could be readily corrected when the expansion of mercury had been redetermined by observations then in progress at the Royal College of Science. These observations have now been extended successfully to a temperature of 300° C. They will be published in full as soon as the final reductions have been made. But as the results exactly confirm the value previously assumed for the boiling point of sulphur, it appeared desirable to remove this uncertainty at the earliest possible date.

So far as we are aware, no serious attempt has been made to redetermine

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* 'Roy. Soc. Proc.,' A, vol. 81, p. 339, 1908.

† 'Roy. Soc. Proc.,' vol. 50, p. 247, 1891.

‡ 'Roy. Soc. Proc.,' A, vol. 81, p. 363, 1908.